

CONVERTING VIRTUAL LINK DIAGRAMS TO NORMAL ONES

NAOKO KAMADA

ABSTRACT. A virtual link diagram is called normal if the associated abstract link diagram is checkerboard colorable, and a virtual link is normal if it has a normal diagram as a representative. In this paper, we introduce a method of converting a virtual link diagram to a normal virtual link diagram by use of the double covering technique. We show that the normal virtual link diagrams obtained from two equivalent virtual link diagrams are related by generalized Reidemeister moves and Kauffman flypes. We obtain a numerical invariant of virtual knots by using our converting method.

1. INTRODUCTION

L. H. Kauffman [8] introduced virtual knot theory, which is a generalization of knot theory based on Gauss diagrams and link diagrams in closed oriented surfaces. Virtual links correspond to stable equivalence classes of links in thickened surfaces [2, 5]. A virtual link diagram is called normal if the associated abstract link diagram is checkerboard colorable (§ 2). A virtual link is called normal if it has a normal diagram as a representative. Every classical link diagram is normal, and hence the set of classical link diagrams is a subset of that of normal virtual link diagrams. The set of normal virtual link diagrams is a subset of that of virtual link diagrams. The f -polynomial (Jones polynomial) is an invariant of a virtual link [8]. It is shown in [3] that the f -polynomial of a normal virtual link has a property that the f -polynomial of a classical link has. This property may make it easier to define Khovanov homology of virtual links as stated in O. Viro [10].

In this paper, we introduce a method of converting a virtual link diagram to a normal virtual link diagram by use of the double covering technique defined in [6]. We show that the normal virtual link diagrams obtained from two equivalent virtual link diagrams by our method are related by generalized Reidemeister moves and Kauffman flypes. We obtain a numerical invariant of virtual knots by using the converting method.

2. DEFINITIONS AND MAIN RESULTS

A *virtual link diagram* is a generically immersed loops with information of positive, negative or virtual crossing, on its double points. A *virtual crossing* is an encircled double point without over-under information [8]. A *twisted link diagram* is a virtual link diagram, possibly with *bars* on arcs. A *virtual link* (or *twisted link*) is an equivalence class of virtual (or twisted) link diagrams under Reidemeister moves and virtual Reidemeister moves (or

This work was supported by JSPS KAKENHI Grant Number 15K04879.

Reidemeister moves, virtual Reidemeister moves and twisted Reidemeister moves) depicted in Figures 1. We call Reidemeister moves and virtual Reidemeister moves *generalized Reidemeister moves*.

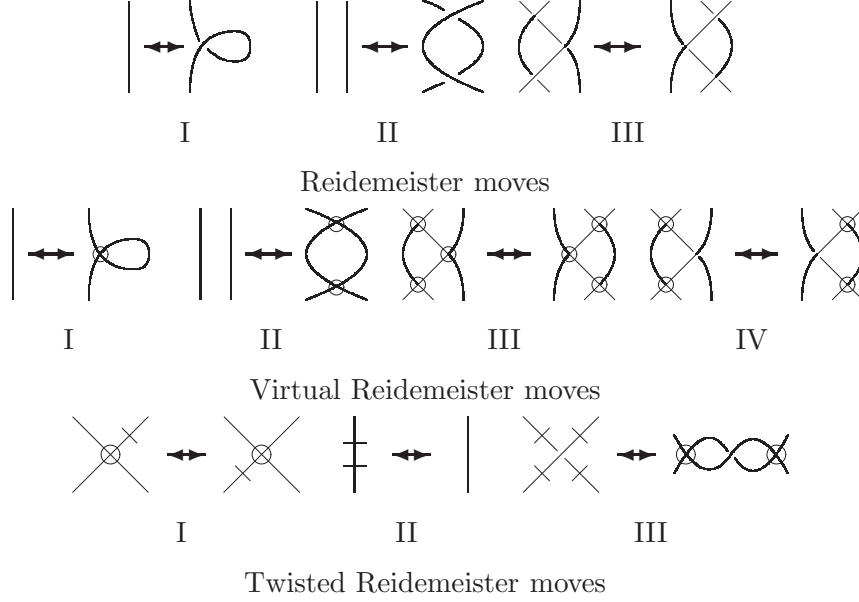


FIGURE 1. Generalized Reidemeister moves and twisted Reidemeister moves

We introduce the double covering diagram of a twisted link diagram. Let D be a twisted link diagram. Assume that D is on the right of the y -axis in the xy -plane and all bars are parallel to the x -axis with disjoint y -coordinates. Let D^* be the twisted link diagram obtained from D by reflection with respect to the y -axis and switching the over-under information of all classical crossings of D . Let $B = \{b_1, \dots, b_k\}$ be a set of bars of D and we denote the bar of D^* corresponding to b_i by b_i^* . See Figure 2 (i). For horizontal lines

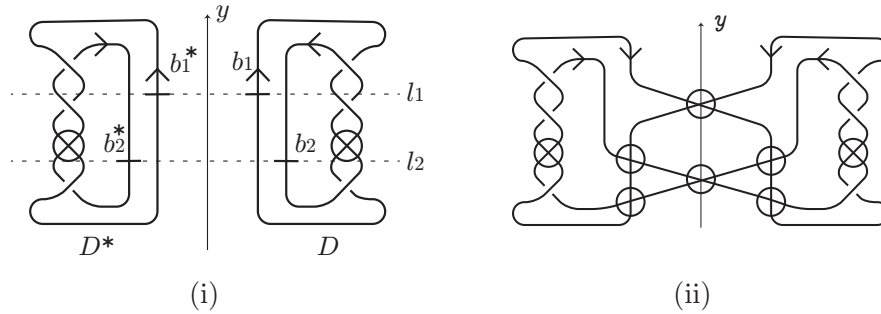


FIGURE 2. The double covering of a twisted link diagram

l_1, \dots, l_k such that l_i contains b_i and the corresponding bar b_i^* of D^* , we replace each part of $D \amalg D^*$ in a neighborhood of $N(l_i)$ for each $i \in \{1, \dots, k\}$ as in Figure 3. We denote by $\phi(D)$ the virtual link diagram obtained this way.

For example, for the twisted link diagram D depicted as in Figure 2 (i), the virtual link diagram $\phi(D)$ is as in Figure 2 (ii). We call this diagram $\phi(D)$ the *double covering*

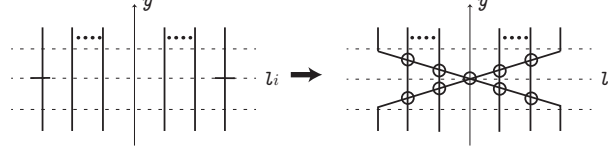


FIGURE 3. The replacement of diagram

diagram of D . Then we have the followings.

Theorem 1. [6] *Let D_1 and D_2 be twisted link diagrams. If D_1 and D_2 are equivalent as a twisted link, then $\phi(D_1)$ and $\phi(D_2)$ are equivalent as a virtual link.*

An *abstract link diagram (ALD)* is a pair of a compact surface Σ and a link diagram D on Σ such that the underlying 4-valent graph $|D|$ is a deformation retract of Σ , denoted by (Σ, D_Σ) .

We obtain an ALD from a twisted link diagram D as in Figure 4. Such an ALD is called the *ALD associated with D* . Figure 5 shows twisted link diagrams and the ALDs associated with them. For details on abstract link diagrams and their relations to virtual links, refer to [5]. Let D be a twisted link diagram and (Σ, D_Σ) the ALD associated with

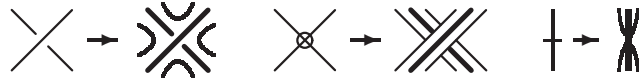


FIGURE 4. The correspondence from a twisted link diagrams to an ALD

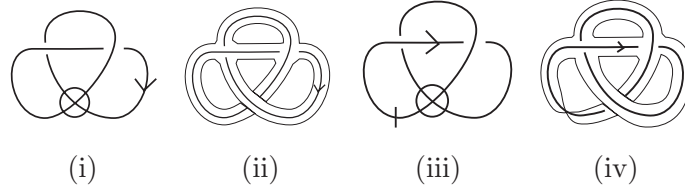


FIGURE 5. Twisted link diagrams and ALDs

D . The diagram D is said to be *normal* or *checkerboard colorable* if the regions of $\Sigma - |D_\Sigma|$ can be colored black and white such that colors of two adjacent regions are different. In Figure 6, we show an example of a normal diagram. A classical link diagram is normal. A twisted link is said to be *normal* if it has a normal twisted link diagram. Note that normality is not necessary to be preserved under generalized Reidemeister moves. For example the virtual link diagram in the right of Figure 7 is not normal and is equivalent to the trefoil knot diagram in the left which is normal.

Proposition 2. *For a normal twisted link diagram D , the double covering diagram of D $\phi(D)$ is normal.*

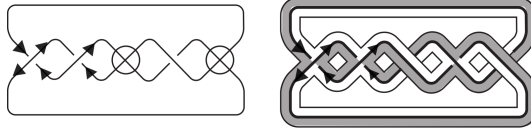


FIGURE 6. A normal twisted link diagram and its associated ALD with a checkerboard coloring

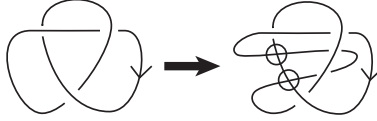


FIGURE 7. A diagram of a normal virtual link which is not normal

Proof. Let D be a normal twisted link diagram D . The twisted link diagram D^* is obtained from D by reflection and switching all classical crossings of D as the previous manner as in Figure 8 (i). The ALDs obtained from D and D^* can be colored as in Figure 8 (ii). Then we see that $\phi(D)$ is normal as in the right of Figure 8 (ii). \square

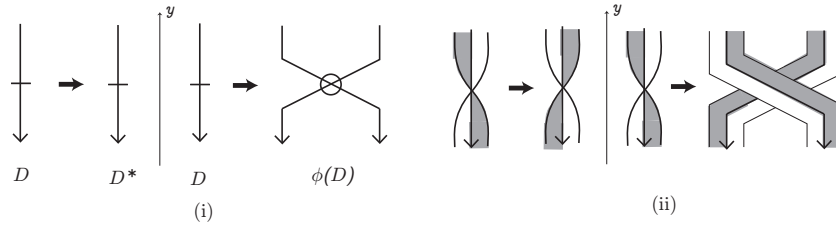


FIGURE 8. The converting diagram

H. Dye introduced the notion of cut points to a virtual link diagram in her talk presented in the Special Session 35, “Low Dimensional Topology and Its Relationships with Physics”, held in Porto, Portugal, June 10-13, 2015 as part of the 1st AMS/EMS/SPM Meeting.

Let (D, P) be a pair of a virtual link diagram D and a finite set P of points on edges of D . We call the ALD associated with the twisted link diagram obtained from (D, P) by replacing all points of P with bars, the *ALD associated with (D, P)* . See Figure 9 (ii) and (iii). If the ALD associated with (D, P) is normal, then we call the set of points P a *cut system* of D and call each point of P a *cut point*. For the virtual link diagram in Figure 9 (i) we show an example of a cut system in Figure 9 (ii) and the ALD associated with it in Figure 9 (iii).

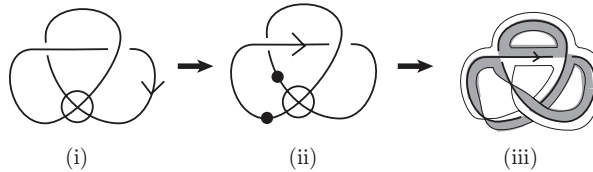


FIGURE 9. Example of cut points

A virtual link diagram is said to admit an *alternate orientation* if it can be given an orientation such that an orientation of an edge switches at each classical crossing as in Figure 10. The virtual link diagram in Figure 6 admits an alternate orientation. It is known that a virtual link diagram is normal (or checkerboard colorable) if and only if it admits an alternate orientation [7].



FIGURE 10. Alternate orientation

Note that a finite set P of points on D is a cut system if and only if (D, P) admits an alternate orientation such that the orientations are as in Figure 11 at each crossing of D and each point of P (cf. [4, 7]).

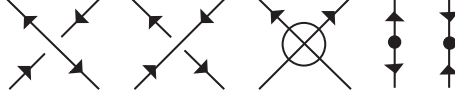


FIGURE 11. Alternate orientation

The *canonical cut system* of a virtual link diagram D is the set of points that is obtained by giving two points in a neighborhood of each virtual crossing of D as in Figure 12 (i).

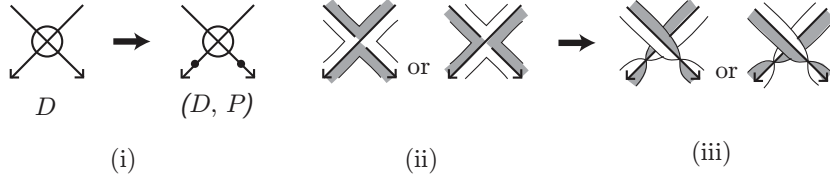


FIGURE 12. Canonical cut system of a virtual link diagram

Proposition 3. *The canonical cut system is a cut system.*

Proof. For a virtual link diagram D , let D_C be a classical link diagram which is obtained from D by replacing all virtual crossings of D with classical ones. Note that there is a checkerboard coloring for the ALD associated with D_C . At each classical crossing, the checkerboard coloring is as in Figure 12 (ii). Let P be the canonical cut system of D . The ALD associated with (D, P) is checkerboard colorable such that its coloring is inherited from that of D_C as in Figure 12 (ii) and (iii). \square

Dye introduced the cut point moves depicted in Figure 13 and asked whether two cut systems of a virtual link diagram D are related by a sequence of cut point moves. The following theorem answers it.

Theorem 4. *For a virtual link diagram D , two cut systems of D are related by a sequence of cut point moves I, II and III.*

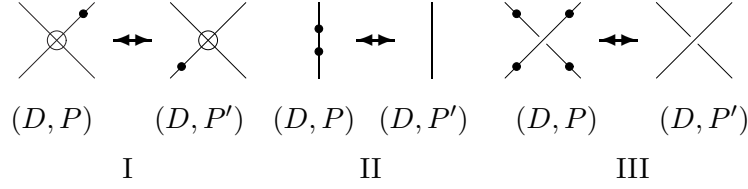


FIGURE 13. The cut point moves

Proof. Let P and P' be two cut systems of D . For (D, P) and (D, P') , give alternate orientations \mathcal{O} and \mathcal{O}' , respectively. Let c_1, c_2, \dots, c_m be classical crossings of D where the orientations of edges of \mathcal{O} are different from those of \mathcal{O}' . Apply cut point moves III at c_1, c_2, \dots, c_m to (D, P) , then we obtain the cut system P'' of D . There is an alternate orientation of (D, P'') , say \mathcal{O}'' , such that each classical crossing in D admits the same orientation to that of \mathcal{O}' . Applying some cut point moves I to P'' , we have a cut system P''' such that for each edge e of D , the number of cut points on e in (D, P''') is congruent to that in (D, P') modulo 2. Then (D, P') is obtained from (D, P''') by cut point moves I and II. \square

Corollary 5 (H. Dye). *For any virtual link diagram with a cut system, the number of cut points is even.*

Proof. The number of cut points of the canonical cut system is even. Since cut point moves do not change the parity of the number of cut points, we obtain the result. \square

Let (D, P) be a virtual link diagram with a cut system. We replace all points of P with bars. Then we obtain a normal twisted link diagram. We denote such a map from the set of virtual link diagrams with cut systems to that of twisted link diagrams by t . We denote the image of (D, P) under t by $t(D, P)$. The double covering of $t(D, P)$ is normal from Proposition 2 since $t(D, P)$ is normal. For a virtual link diagram with a cut system (D, P) the double covering diagram of $t(D, P)$ is called the *converted normal diagram* of (D, P) , denoted by $\phi(D, P)$.

The local replacement of a virtual link diagram depicted in Figure 14 is called a *Kauffman flype* or a *K-flype*. If a virtual link diagram D' is obtained from D by a finite sequence of generalized Reidemeister moves and K-flypes, then they are said to be *K-equivalent*.

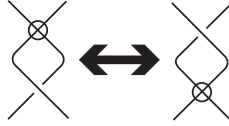


FIGURE 14. Kauffman flype

Remark 6. The f -polynomials of K-equivalent virtual link diagrams are the same [8]. For a virtual link diagram of D , if a virtual link diagram D' is obtained from D by a K-flype at a classical crossing c , then the sign of the corresponding classical crossing c' of D' is the same as that of c . If D is normal, then D' is normal.

The following is our main theorem.

Theorem 7. *Let (D, P) and (D', P') be virtual link diagrams with cut systems. If D' is equivalent (or K -equivalent) to D , then the converted normal diagram $\phi(D, P)$ is K -equivalent to $\phi(D', P')$.*

For a 2-component of virtual link diagram D , the half of the sum of signs of non self-crossings of D is said to be the *linking number* of D . The following is clear.

Proposition 8. *The linking number is invariant under the generalized Reidemeister moves and K -flypes.*

We have the following theorems from our main theorem and Proposition 8.

Theorem 9. *Let (D, P) be a virtual knot diagram with a cut system. Then $\phi(D, P)$ is a 2-component virtual link diagram and the linking number of $\phi(D, P)$ is an invariant of the virtual knot represented by D .*

The odd writhe is a numerical invariant of virtual knots [9]. We recall the definition of the odd writhe in Section 4.

Theorem 10. *Let (D, P) be a virtual knot diagram with a cut system. The linking number of $\phi(D, P)$ coincides to the odd writhe of D .*

3. PROOF OF THEOREM 7

Theorem 7 is obtained from Lemmas 11 and 12 stated below.

Lemma 11. *Let D be a virtual link diagram. Suppose that P and P' are two cut systems of D . Then the converted normal diagrams $\phi(D, P)$ and $\phi(D, P')$ are K -equivalent.*

Proof. Let D be a virtual link diagram with a cut system P . Suppose that P' is a cut system of D obtained from P by one of cut point moves I or II in Figure 13. Then $t(D, P')$ is obtained from $t(D, P)$ by a twisted Reidemeister move I or II, respectively. Thus by Theorem 1, $\phi(D, P)$ and $\phi(D, P')$ are equivalent. If P' is related to P by a cut point move III in Figure 13, then $\phi(D, P)$ and $\phi(D, P')$ are related by virtual Reidemeister moves and K -flypes as in Figure 15. If the orientations of some strings of a virtual link diagram are different from those in Figure 15, we have the result by a similar argument. \square

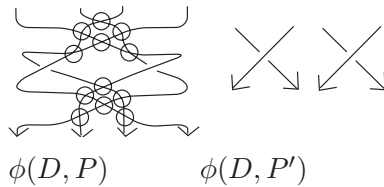


FIGURE 15. Converted normal diagrams related by a cut point move III

Lemma 12. *Let (D, P) and (D', P') be virtual link diagrams with canonical cut systems. If D' is equivalent (or K -equivalent) to D , then $\phi(D, P)$ and $\phi(D', P')$ are K -equivalent.*

Proof. Let D_1 be a virtual link diagram with the canonical cut system P_1 . Suppose that a virtual link diagram D_2 is obtained from D_1 by one of generalized Reidemeister moves or K -flype and P_2 is the canonical cut system of D_2 . If D_2 is related to D_1 by one of Reidemeister moves, then $\phi(D_1, P_1)$ and $\phi(D_2, P_2)$ are related by two Reidemeister moves. As in Figure 16 (i) or (ii), suppose that D_2 is related to D_1 by a virtual Reidemeister move I or II and let P'_2 be the cut system obtained from P_2 by cut point moves I and II as in the figure. By Lemme 11, $\phi(D_2, P_2)$ and $\phi(D_2, P'_2)$ are equivalent. By Theorem 1, $\phi(D_1, P_1)$ and $\phi(D_2, P'_2)$ are equivalent since $t(D_1, P_1)$ and $t(D_2, P'_2)$ are related by virtual Reidemeister move I or II, which means that $\phi(D_1, P_1)$ and $\phi(D_2, P_2)$ are equivalent. As in Figure 16 (iii), suppose that D_2 is related to D_1 by a virtual Reidemeister move III and let P'_1 (or P'_2) be the cut system obtained from P_1 (or P_2) by cut point moves I and II as in the figure. By Lemme 11, $\phi(D_1, P_1)$ (or $\phi(D_2, P_2)$) and $\phi(D_1, P'_1)$ (or $\phi(D_2, P'_2)$) are equivalent. By Theorem 1 $\phi(D_1, P'_1)$ and $\phi(D_2, P'_2)$ are equivalent since $t(D_1, P'_1)$ and $t(D_2, P'_2)$ are related by virtual Reidemeister move III, which means that $\phi(D_1, P_1)$ and $\phi(D_2, P_2)$ are equivalent. As in Figure 16 (iv), suppose that D_2 is related to D_1 by a virtual Reidemeister move IV and let P'_1 (or P'_2) be the cut system obtained from P_1 (or P_2) by cut point moves I and II (or cut point moves I, II and III) as in the figure. By the similar reason, $\phi(D_1, P_1)$ and $\phi(D_2, P_2)$ are K -equivalent. If D_1 is related to D_2 by a

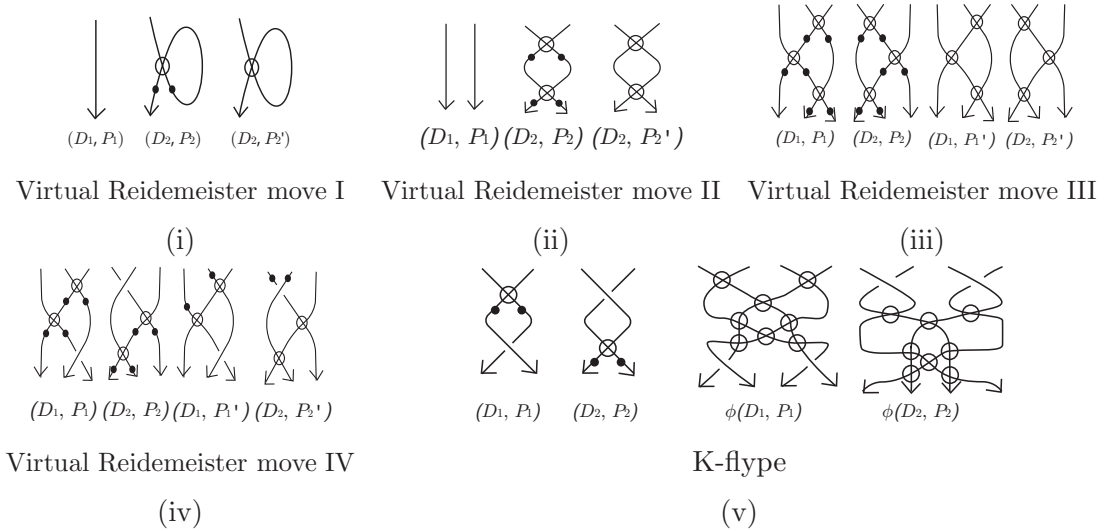


FIGURE 16. Diagrams related by a virtual Reidemeister moves and a K -flype

K -flype, then $\phi(D_1, P_1)$ and $\phi(D_2, P_2)$ are related by some virtual Reidemeister moves in Figure 16 (v). In Figure 16, if the orientation of some strings of virtual link diagram D_i are different from those of it, we have the result by a similar argument. \square

4. PROOF OF THEOREMS 9 AND 10

Let D be a virtual link diagram. The *Gauss diagram* of D is a set of oriented circles such that each component is the preimage of D with oriented chords each of which corresponds to a classical crossing and its starting point (or ending point) indicates an over path (or an under path) of the classical crossing. Each chord is equipped with a sign of the corresponding classical crossing. The Gauss diagram in Figure 17 (i) is that of the virtual knot diagram in Figure 5 (i). For a virtual link diagram D with a set of points, P , the

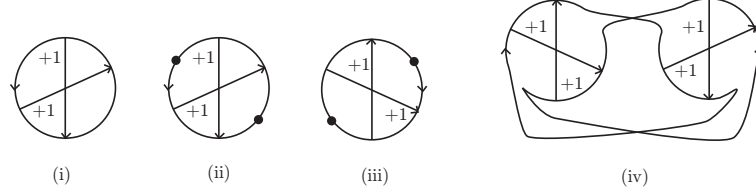


FIGURE 17. The Gauss diagrams

Gauss diagram with points of (D, P) is obtained from the Gauss diagram of D by adding points on arcs which correspond to the points of D . In Figure 17 (ii) we see the Gauss diagram with a set of points of the virtual link diagram with a set of points in Figure 9 (ii). We denote the Gauss diagram of D with a set of points, P by $G(D, P)$. Let (D^*, P^*) be the virtual link diagram with a cut system which is obtained from (D, P) by reflection with respect to y -axis and switching the over-under information of all classical crossings of D . The Gauss diagram with points of (D^*, P^*) is obtained from the Gauss diagram of (D, P) by reflection with respect to y -axis and revering all orientations of chords. For example, the Gauss diagram with points in Figure 17 (iii) is the Gauss diagram of the virtual knot diagram with a set of points obtained from a virtual knot diagram in Figure 5 (ii) by the reflection with respect to y -axis and switching all classical crossings. The Gauss diagram of $\phi(D, P)$ is obtained from the Gauss diagrams of $D \amalg D^*$ by a local replacement around each point p of the Gauss diagram of D and around the corresponding point p^* of that of D^* as in Figure 18. For a virtual knot diagram of D with a set of points on edges P , we denote the Gauss diagram of $\phi(D, P)$ by $G(\phi(D, P))$. The Gauss diagram in Figure 17 (iv) is $G(\phi(D, P))$ for the (D, P) depicted in Figure 9 (ii) by the map ϕ . For the Gauss

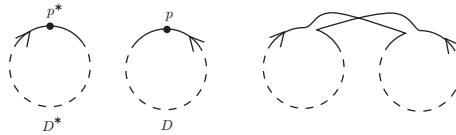


FIGURE 18. Replacement of Gauss diagrams

diagram of $(D \amalg D^*, P \amalg P^*)$, suppose that p_1, \dots, p_n are points in P such that the point p_{i+1} follows the point p_i along the orientation of D and the point p_i^* in P^* is symmetric to the point p_i . Let A_i (or A_i^*) be the arc of the Gauss diagram of $G(D \amalg D^*, P \amalg P^*)$

between two points p_i and p_{i+1} (or p_i^* and p_{i+1}^*), and the arc between two points p_n and p_1 (or p_n^* and p_1^*) is A_n (or A_n^*). Note that A_i is symmetric to A_i^* . We also denote an arc of $G(\phi(D, P))$ which corresponds to A_i or A_i^* of $G(D \amalg D^*, P \amalg P^*)$ by $\widetilde{A_i}$ or $\widetilde{A_i^*}$, respectively. Here $\widetilde{A_i}$ (or $\widetilde{A_i^*}$) is the arc in $G(\phi(D, P))$ which is obtained from A_i (or A_i^*) by removing a regular neighborhood of p_i and p_{i+1} (or p_i^* and p_{i+1}^*).

We proof the following lemmas in stead of Theorem 9.

Lemma 13. *Let D be a virtual knot diagram with a set of points on edges $P = \{p_1, \dots, p_{2n}\}$ for a positive integer n . Then $\phi(D, P)$ is a 2-component virtual link diagram $D_1 \cup D_2$. Furthermore if an arc A_i belongs to $G(\phi(D, P))|_{D_1}$ (or $G(\phi(D, P))|_{D_2}$), then A_{i+1} and A_i^* belong to $G(\phi(D, P))|_{D_2}$ (or $G(\phi(D, P))|_{D_1}$).*

Proof. We use the induction on n . Suppose that $n = 1$, i.e., D is a virtual knot diagram with 2 points p_1 and p_2 . The Gauss diagram $G(\phi(D, P))$ is depicted as in Figure 19, where the bold line and the thin line indicate the different components and we dropped all chords in the figure. In this case $\phi(D, P)$ is a 2-component virtual link diagram. Two arcs $\widetilde{A_1}$ and $\widetilde{A_2^*}$ are in one component of $G(\phi(D, P))$, and $\widetilde{A_2}$ and $\widetilde{A_1^*}$ are in the other. Suppose that the statement is hold if the number of points is less than $2n$. We assume

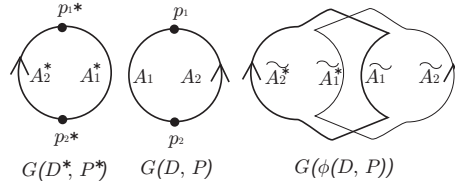


FIGURE 19. $G(\phi(D, P))$ of D a pair of point

that D is a virtual knot diagram with $2n$ points p_1, \dots, p_{2n} . We apply the replacement as in Figure 18 to $2n - 2$ points p_1, \dots, p_{2n-2} and p_1^*, \dots, p_{2n-2}^* . Then we obtain the Gauss diagram G with 4 points p_{2n-1}, p_{2n} and p_{2n-1}^*, p_{2n}^* . By the hypothesis, the Gauss diagram G is depicted as in Figure 20 (i), where two arcs A_{2n-1} and A_{2n} (or two points p_{2n-1} and p_{2n}) are in one component of G and two arcs A_{2n-1}^* and A_{2n}^* (or two points p_{2n-1}^* and p_{2n}^*) are in the other. If an arc $\widetilde{A_i}$ is in one component of G , $\widetilde{A_i^*}$ (or $\widetilde{A_{i+1}}$) is in the other for $i \neq 2n - 1$ by the hypothesis. By applying the replacement in Figure 18 to two pairs of points p_{2n-1} and p_{2n-1}^* and p_{2n} and p_{2n}^* of the Gauss diagram G , we have the Gauss diagram as in Figure 20 (ii). Therefore we have the result. \square

Thus we have the following lemma

Lemma 14. *Let (D_1, P_1) and (D_2, P_2) be virtual knot diagrams with cut systems. If D_1 and D_2 are equivalent (or K -equivalent), then the linking number of $\phi(D_2, P_2)$ is equal to that of $\phi(D_1, P_1)$.*

Proof. By Theorem 7, $\phi(D_1, P_1)$ and $\phi(D_2, P_2)$ are K -equivalent. Hence by Proposition 8, they have the same linking numbers. \square

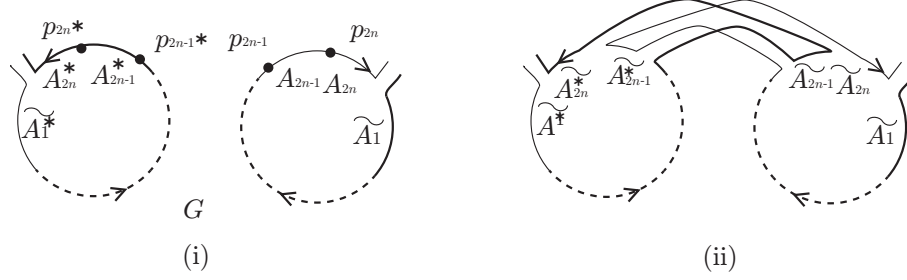


FIGURE 20. Replacement of Gauss diagrams

For a virtual knot K and its diagram D with a cut system P , we denote the linking number of $\phi(D, P)$ by $\text{lk}_N(K)$ or $\text{lk}_N(D)$. It does not depend on the choice of P by Lemma 14.

Let D be a virtual knot diagram and G be a Gauss diagram of D . For a classical crossing c , we denote by γ_c the chord of G corresponding to c . The endpoints of γ_c divide the circle of G into 2 arcs. We denote the arcs by I_c and I'_c where I_c is the arc which starts from the tail of γ_c and terminates at the head. A classical crossing c of D is said to be *odd* if there are an odd number of endpoints of chords of G on I_c . The *odd writhe* of D is the sum of signs of odd crossings of D . Note that if a virtual knot diagram is normal, all classical crossings are not odd.

Theorem 15 ([9]). *The odd writhe is an invariant of virtual knots.*

Proof of Theorem 10. Let D be a virtual knot diagram and P be a cut system of D . It is sufficient to show that odd crossings of D correspond to non self classical crossings of $\phi(D, P)$. Since (D, P) admits an alternate orientation, the circle of the Gauss diagram $G(D, P)$ of (D, P) admits an alternate orientation such that one endpoint of each chord is a sink of the orientations and the other is a source. This implies the following condition:

- (*) For any classical crossing c of (D, P) , the sum of the number of cut points and that of endpoints of chords appearing on the arc I_c , is even.

Let A_1, A_2, \dots be the arcs obtained by cutting the circle of $G(D, P)$ along the cut points. We assume that the arc A_{i+1} appears after the arc A_i along the orientation of D . We also denote an arc of $G(\phi(D, P))|_D$ which corresponds to A_i by \widetilde{A}_i . For a classical crossing c of (D, P) , the classical crossing corresponding to c in $\phi(D, P)|_D$ is denoted by \tilde{c} . Let c be an odd crossing of (D, P) . Suppose that one endpoint of γ_c is on A_k and the other endpoint is on A_j . By definition, there are an odd number of endpoints of chords on I_c in $G(D, P)$. By the condition (*) above, we see that there are an odd number of cut points on I_c in $G(D, P)$. Then we see that k is not congruent to j modulo 2. By Lemma 13 \widetilde{A}_k and \widetilde{A}_j in $G(\phi(D, P))|_D$ are in the different components of $G(\phi(D, P))$. Thus the classical crossing \tilde{c} is a non self classical crossing of $\phi(D, P)$. \square

We show some property of $\text{lk}_N(D)$. They are also obtained from the property of the odd writhe.

Corollary 16 ([9]). *Let K be a normal virtual knot. Then $\text{lk}_N(K)$ is zero.*

Proof. Let D_N be a normal knot diagram of K . Any cut system of D_N is related to an empty set by some cut point moves since D_N is normal. The virtual link diagram $\phi(D_N, \emptyset)$ is the disjoint union of D_N and D_N^* , $D_N \amalg D_N^*$. Thus we see that $\text{lk}_N(K)$ is zero by Proposition 8, since the linking number of $D_N \amalg D_N^*$ is zero. \square

Let D be a virtual knot diagram. The virtual knot diagram obtained from D by switching the over-under information of all classical crossing (or by reflection) is denoted by D^\sharp (or D^\dagger).

Corollary 17 ([9]). *Let D be a virtual knot diagram. If $\text{lk}_N(D)$ is not zero, then D is not equivalent to D^\sharp (or D^\dagger).*

Proof. It is clear that $\text{lk}_N(D^\sharp) = \text{lk}_N(D^\dagger) = -\text{lk}_N(D)$. \square

For example, the virtual knot presented by the diagram D in Figure 21, is not normal by Corollary 16, since $\text{lk}_N(D) = -2$. By Corollary 17, D is not equivalent to D^\sharp (or D^\dagger).

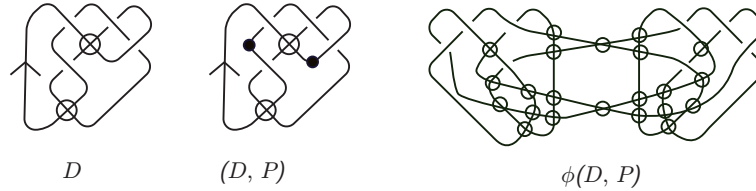


FIGURE 21. A non normal virtual link

Acknowledgement

The author would like to thank Seiichi Kamada for his useful suggestion.

REFERENCES

- [1] M. O. Bourgoïn, *Twisted link theory*, Algebr. Geom. Topol. 8 (2008) 1249–1279.
- [2] J. S. Carter, S. Kamada and M. Saito, *Stable equivalence of knots on surfaces and virtual knot cobordisms*, J. Knot Theory Ramifications 11 (2002) 311–322.
- [3] N. Kamada, *On the Jones polynomials of checkerboard colorable virtual knots*, Osaka J Math., 39 (2002), 325–333.
- [4] N. Kamada, *On twisted knots*, Contemp. Math., to appear.
- [5] N. Kamada and S. Kamada, *Abstract link diagrams and virtual knots*, J. Knot Theory Ramifications 9 (2000) 93–106.
- [6] N. Kamada and S. Kamada, *Double coverings of twisted links*, preprint (arXiv:1510.03001).
- [7] N. Kamada, S. Nakabo and S. Satoh, *A virtualized skein relation for Jones polynomials*, Illinois J. Math. 46 (2002), 467–475.
- [8] L. H. Kauffman, *Virtual knot theory*, European J. Combin. 20 (1999) 663–690.
- [9] L. H. Kauffman, *A self-linking invariant of virtual knots*. Fund. Math., 184 (2004), 135–158.
- [10] O. Viro, *Virtual Links, Orientations of Chord Diagrams and Khovanov Homology*, Proceedings of 12th Gökova Geometry-Topology Conference, (2007) 184–209.

GRADUATE SCHOOL OF NATURAL SCIENCES, NAGOYA CITY UNIVERSITY, 1 YAMANOHATA, MIZUHO-CHO, MIZUHO-KU, NAGOYA, AICHI 467-8501 JAPAN